

# Electromagnetic wave propagation inside a material medium: an effective geometry interpretation

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## Abstract

We present a method developed to deal with electromagnetic wave propagation inside a material medium that reacts, in general, non-linearly to the field strength. We work in the context of Maxwell's theory in the low frequency limit and obtain a geometrical representation of light paths for each case presented. The isotropic case and artificial birefringence caused by an external electric field are analyzed as an application of the formalism and the effective geometry associated to the wave propagation is exhibited.

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## I. INTRODUCTION

The characteristics of electromagnetic waves propagating in vacuum are governed by a second order partial differential equation. However, when propagating through dispersive media, such differential equation is no longer satisfied and, in general, it is necessary to deal with a complicated system of equations. As a consequence, several phenomena have to be analyzed closer, like polarization states, wave velocity and its paths inside the particular medium where it is traveling. Indeed, such deviation from the linear regime is also found to occur in vacuum when the field strength goes beyond its critical value ( $E_c \sim B_c = m^2 c^2 / e \hbar$ ), which is the limit of applicability [1] of the linear Maxwell's theory. The theoretical description of light propagation in nonlinear electrodynamics was first presented by Bialynicka-Birula and Bialynicki-Birula [2] and also by Adler [3], where the probability of the photon splitting under a strong external electromagnetic field was studied. Dittrich and Gies [4] obtained the light cone conditions for a class of homogeneous non trivial QED vacua by using the rule of average over polarization states. More recently De Lorenci, Klippert, Novello and Salim [5] studied the same problem within a different formalism, in which general light cone conditions for a class of theories constructed with the two gauge invariants ( $F, G$ ) of the Maxwell field was derived without making use of average over polarization states. There, birefringence phenomena [6] is also described. Further, the same authors [7] analyzed the geometrical aspects of wave propagation in the context of nonlinear electrodynamics and found that the paths of light can be described in terms of an effective geometry which represents a modification of the Minkowski metric. It depends on the dynamics of the background electromagnetic field as

$$g_{\pm}^{\mu\nu} = \eta^{\mu\nu} - \frac{4}{L_F} \left[ [(L_{FF} + \Omega_{\pm} L_{FG}) F^{\mu}_{\lambda} F^{\lambda\nu} + (L_{FG} + \Omega_{\pm} L_{GG}) F^{\mu}_{\lambda} \tilde{F}^{\lambda\nu}] \right] \quad (1)$$

where  $\Omega_{\pm}$  are certain coefficients depending on the dynamics and field strength, and  $L_X$  represents the derivative of the Lagrangian with respect to the invariant  $X = (F, G)$ . Indeed, it is shown that photons described by nonlinear electrodynamics propagate as null geodesics in such effective metric. (See also Ref. [8] where the effective geometry interpretation is considered in the analysis of the Scharnhorst effect.) Therefore, applying the previous formalism for material media is a difficult task, since for most cases we do not know an equivalent Lagrangian expression describing the physical system. Inside such media, Maxwell's equations are supplemented by the constitutive law that relates the electromagnetic excitations  $D, B$  and field strengths  $E, H$  with the quantities describing the electric and magnetic properties of each medium. The constitutive equations are, in general, nonlinear and are determined phenomenologically. Schönberg [9] and later Obukhov and Hehl [10] derived the equivalent spacetime metric for Maxwell's theory on an arbitrary (1+3)-dimensional manifold imposing a linear constitutive law. Within the same approach, Obukhov, Fukui and Rubilar [11] investigated the wave propagation in the Maxwell electrodynamics with the most general linear constitutive law and derived the associated Fresnel equation which determines the wave normals from the constitutive coefficients.

In this paper we analyze the electromagnetic wave propagation in material media in the geometrical optics approximation. We present the general field equations in terms of some functions depending on the properties of the medium

without making any restriction to the constitutive law. The geometrical representation of the light paths for some special cases are examined and the effective metric associated to each situation are exhibited.

In order to achieve more simplicity, in what follows we work in Minkowski spacetime employing a Cartesian coordinate system. The background metric will be represented by  $\eta_{\mu\nu}$ , which is defined by  $diag(+1, -1, -1, -1)$ . We set the units  $c = 1$ .

## II. THE FIELD EQUATIONS

We define the antisymmetric tensors  $F_{\mu\nu}$  and  $P_{\mu\nu}$  representing the total electromagnetic field. They can be expressed in terms of the strengths  $(E, H)$  and the excitations  $(D, B)$  of the electric and magnetic fields as

$$F_{\mu\nu} = V_\mu E_\nu - V_\nu E_\mu - \eta_{\mu\nu}^{\alpha\beta} V_\alpha B_\beta \quad (2)$$

$$P_{\mu\nu} = V_\mu D_\nu - V_\nu D_\mu - \eta_{\mu\nu}^{\alpha\beta} V_\alpha H_\beta \quad (3)$$

where  $V_\mu$  represents the velocity 4-vector of an arbitrary observer, which in Galilean coordinates will be given by  $V_\mu = \delta_\mu^0$ . The Levi-Civita tensor introduced above is defined in such way that  $\eta^{0123} = +1$ . Since the electric and magnetic tensor fields contain only spatial components, we will denote the products as  $E^\alpha E_\alpha = -E^2$  and  $H^\alpha H_\alpha = -H^2$ . In general the properties of the media are determined by the tensors  $\varepsilon_{\alpha\beta}$  and  $\mu_{\alpha\beta}$  which relate the electromagnetic excitation and the field strength by the generalized constitutive laws,

$$D_\alpha = \varepsilon_\alpha^\beta (E^\mu, H^\mu) E_\beta \quad (4)$$

$$B_\alpha = \mu_\alpha^\beta (E^\mu, H^\mu) H_\beta. \quad (5)$$

In the absence of sources Maxwell's theory can be summarized by the equations

$$V^\mu D^\nu{}_{,\nu} - V^\nu D^\mu{}_{,\nu} - \eta^{\mu\nu\alpha\beta} V_\alpha H_{\beta,\nu} = 0 \quad (6)$$

$$V^\mu B^\nu{}_{,\nu} - V^\nu B^\mu{}_{,\nu} + \eta^{\mu\nu\alpha\beta} V_\alpha E_{\beta,\nu} = 0 \quad (7)$$

which is equivalent to  $P^{\mu\nu}{}_{,\nu} = 0$  and  $\overset{*}{F}{}^{\mu\nu}{}_{,\nu} = 0$ , respectively. Therefore, the electromagnetic excitation is related to the field strength by means of the constitutive relations (4) and (5), whose derivatives with respect to the coordinates can be presented as

$$D_{\alpha,\tau} = \varepsilon_\alpha^\beta E_{\beta,\tau} + \frac{\partial \varepsilon_\alpha^\beta}{\partial E_\mu} E_\beta E_{\mu,\tau} + \frac{\partial \varepsilon_\alpha^\beta}{\partial H_\mu} E_\beta H_{\mu,\tau} \quad (8)$$

$$B_{\alpha,\tau} = \mu_\alpha^\beta H_{\beta,\tau} + \frac{\partial \mu_\alpha^\beta}{\partial E_\mu} H_\beta E_{\mu,\tau} + \frac{\partial \mu_\alpha^\beta}{\partial H_\mu} H_\beta H_{\mu,\tau}. \quad (9)$$

Equations (6) and (7), together with the relations (8) and (9), are given in terms of the first derivatives of the electric and magnetic fields. They represent the electromagnetic field equations inside an arbitrary material medium whose properties are determined by the tensors  $\varepsilon_{\alpha\beta}$  and  $\mu_{\alpha\beta}$ .

## III. PROPAGATING WAVES

In order to derive the relations that determine the propagating waves, we will consider the Hadamard method of field discontinuities [5,12]. Let us consider a surface of discontinuity  $\Sigma$  defined by  $z(x^\mu) = 0$ . Whenever  $\Sigma$  is a global surface, it divides the spacetime in two distinct regions  $U^-$ , for  $z < 0$ , and  $U^+$ , for  $z > 0$ . Given an arbitrary function of the coordinates,  $f(x^\alpha)$ , we define its discontinuity on  $\Sigma$  as

$$[f(x^\alpha)]_\Sigma \doteq \lim_{\{P^\pm\} \rightarrow P} [f(P^+) - f(P^-)] \quad (10)$$

where  $P^+$ ,  $P^-$  and  $P$  belong to  $U^+$ ,  $U^-$  and  $\Sigma$  respectively. Applying these conditions for the electric and magnetic fields and their derivatives we set

$$[E_\mu]_\Sigma = 0; \quad [E_{\mu,\nu}]_\Sigma = e_\mu K_\nu \quad (11)$$

$$[H_\mu]_\Sigma = 0; \quad [H_{\mu,\nu}]_\Sigma = h_\mu K_\nu \quad (12)$$

where  $e_\mu$  and  $h_\mu$  represent the discontinuities of the fields on the surface  $\Sigma$  and  $K_\lambda$  is the wave propagation 4-vector. Applying these conditions to the field equations (6) and (7), we obtain the following set of equations governing the wave propagation

$$\varepsilon^{\alpha\beta} K_\alpha e_\beta + \frac{\partial \varepsilon^{\alpha\beta}}{\partial E_\mu} K_\alpha E_\beta e_\mu + \frac{\partial \varepsilon^{\alpha\beta}}{\partial H_\mu} K_\alpha E_\beta h_\mu = 0 \quad (13)$$

$$\mu^{\alpha\beta} K_\alpha h_\beta + \frac{\partial \mu^{\alpha\beta}}{\partial E_\mu} K_\alpha H_\beta e_\mu + \frac{\partial \mu^{\alpha\beta}}{\partial H_\mu} K_\alpha H_\beta h_\mu = 0 \quad (14)$$

$$\left( \varepsilon^{\mu\beta} e_\beta + \frac{\partial \varepsilon^{\mu\beta}}{\partial E_\alpha} E_\beta e_\alpha + \frac{\partial \varepsilon^{\mu\beta}}{\partial H_\alpha} E_\beta h_\alpha \right) (KV) + \eta^{\mu\nu\alpha\beta} K_\nu V_\alpha h_\beta = 0 \quad (15)$$

$$\left( \mu^{\mu\beta} h_\beta + \frac{\partial \mu^{\mu\beta}}{\partial E_\alpha} H_\beta e_\alpha + \frac{\partial \mu^{\mu\beta}}{\partial H_\alpha} H_\beta h_\alpha \right) (KV) - \eta^{\mu\nu\alpha\beta} K_\nu V_\alpha e_\beta = 0 \quad (16)$$

where we have defined  $(KV) \doteq K^\mu V_\mu$ . Indeed, equations (13) and (14) follow from the zeroth component of equations (6) and (7), respectively.

We will restrict our analysis for those cases where  $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(E^\mu, H^\nu)$  and  $\mu_{\alpha\beta} = \mu\eta_{\alpha\beta}$ . (From the symmetry of the field equations, all results we obtain are equally true for the tensor  $\mu_{\alpha\beta}$ ). Thus, from equation (14) results the identity  $h^\beta K_\beta = 0$ , and equation (16) gives the following relation between  $h_\alpha$  and  $e_\alpha$ :

$$h^\mu = \frac{1}{\mu(KV)} \eta^{\mu\nu\alpha\beta} K_\nu V_\alpha e_\beta. \quad (17)$$

By introducing these results in equation (15) it follows that

$$Z^{\mu\beta} e_\beta = 0 \quad (18)$$

with

$$Z^{\mu\beta} = \mu(KV)^2 \left( \varepsilon^{\mu\beta} + \frac{\partial \varepsilon^{\mu\alpha}}{\partial E_\beta} E_\alpha \right) + (KV) \frac{\partial \varepsilon^{\mu\rho}}{\partial H_\alpha} \eta_{\alpha}^{\tau\sigma\beta} E_\rho K_\tau V_\sigma - K^\mu K^\beta + (KV) V^\mu K^\beta + K^2 \eta^{\mu\beta} - (KV)^2 \eta^{\mu\beta}. \quad (19)$$

Non-trivial solutions of equation (18) can be found for such cases where  $\det |Z^{\mu\beta}| = 0$ . This condition is known in the literature as the Fresnel equation. Despite of its importance, we will not be interested in solving it explicitly, but in analyzing some special cases for which an effective metric structure associated to the wave propagation can be derived. For the cases where the constitutive laws are linear, the Fresnel equation was solved in Ref. [10,11].

#### IV. THE EFFECTIVE GEOMETRY FOR MEDIA WITH ISOTROPIC PERMITTIVITY

The special case of media with isotropic permittivity can be stated by assuming  $\varepsilon^{\mu\beta} = \epsilon(E, H)(\delta^{\mu\beta} - V^\mu V^\beta)$ . Introducing this condition in equation (18), we obtain

$$\left\{ [K^2 - (1 - \mu\epsilon)(KV)^2] \eta^{\mu\beta} - (KV)^2 \frac{\mu}{E} \frac{\partial \epsilon}{\partial E} E^\mu E^\beta - (KV) \frac{1}{H} \frac{\partial \epsilon}{\partial H} \eta^{\tau\nu\alpha\beta} H_\tau K_\nu V_\alpha E^\mu - K^\mu K^\beta + (KV) V^\mu K^\beta \right\} e_\beta = 0. \quad (20)$$

This equation can be solved by expanding  $e_\beta$  as a linear combination of four linearly independent vectors, which can be chosen to be  $E_\beta$ ,  $H_\beta$ ,  $K_\beta$  and  $V_\beta$  as  $e_\beta = \alpha_0 E_\beta + \beta_0 H_\beta + \gamma_0 K_\beta + \delta_0 V_\beta$ . Thus, substituting such representation of  $e_\beta$  in equation (20) and considering the linear independence of the vectors used to expand it, it yields the following equations

$$\alpha_0 \left[ \frac{K^2}{(KV)^2} - 1 + \mu \frac{\partial(\epsilon E)}{\partial E} - \frac{1}{(KV)H} \frac{\partial \epsilon}{\partial H} \eta^{\tau\nu\alpha\beta} H_\tau K_\nu V_\alpha E_\beta \right] - \beta_0 \left[ \frac{\mu}{E} \frac{\partial \epsilon}{\partial E} E^\alpha H_\alpha \right] - \gamma_0 \left[ \frac{\mu}{E} \frac{\partial \epsilon}{\partial E} E^\alpha K_\alpha \right] = 0 \quad (21)$$

$$\alpha_0 E^\mu K_\mu + \beta_0 H^\mu K_\mu + \gamma_0 (1 - \mu\epsilon)(KV)^2 + \delta_0 (KV) = 0 \quad (22)$$

$$\alpha_0 (KV) E^\mu K_\mu + \beta_0 (KV) H^\mu K_\mu + \gamma_0 (KV) K^2 + \delta_0 [K^2 + \mu\epsilon(KV)^2] = 0 \quad (23)$$

$$\beta_0 [K^2 - (1 - \mu\epsilon)(KV)^2] = 0. \quad (24)$$

The solution of this system results in the light cone condition

$$K^2 = (KV)^2 \left[ 1 - \mu \frac{\partial(\epsilon E)}{\partial E} \right] + \frac{1}{\epsilon E} \frac{\partial \epsilon}{\partial E} E^\alpha E^\beta K_\alpha K_\beta + (KV) \frac{1}{H} \frac{\partial \epsilon}{\partial H} \eta^{\tau\nu\alpha\beta} H_\tau K_\nu V_\alpha E_\beta. \quad (25)$$

The wave propagation set by equation (25) can be presented as  $g^{\mu\nu} K_\mu K_\nu = 0$ , where we have defined the effective geometry

$$g^{\mu\nu} = \eta^{\mu\nu} - \left[ 1 - \mu \frac{\partial(\epsilon E)}{\partial E} \right] V^\mu V^\nu - \frac{1}{\epsilon E} \frac{\partial \epsilon}{\partial E} E^\mu E^\nu - \frac{1}{2H} \frac{\partial \epsilon}{\partial H} \eta^{\tau\alpha\beta(\mu} V^{\nu)} H_\tau V_\alpha E_\beta \quad (26)$$

in which the notation  $(\mu\nu)$  designates the operation of symmetrization over the index, e.g.  $a^{(\mu\nu)} = a^{\mu\nu} + a^{\nu\mu}$ . In this way we see that  $K_\mu$  is a null vector in the effective geometry  $g_{\mu\nu}$  which constitutes a deviation from Minkowski background due to the dielectric media in which the wave propagates. Indeed, it can be shown [7] that  $K_\mu$  satisfies the geodesic equation  $K_{\mu;\lambda} K^\lambda = 0$ , where the semicolon designates the covariant derivative in the geometry determined by  $g_{\mu\nu}$ . Thus, since  $K_\mu$  is a null vector in such effective geometry, the integral curves of  $K_\mu$  will be null geodesics.

The phase velocity  $v$  of the waves can be derived from the light cone condition (25) by imposing  $K^2 = \omega^2 - |\vec{K}|^2$  and defining  $v = \omega/|\vec{K}|$ , resulting

$$v^2 = \left[ \mu \frac{\partial(\epsilon E)}{\partial E} - \frac{1}{\omega H} \frac{\partial \epsilon}{\partial H} \vec{K} \cdot \vec{E} \times \vec{H} \right]^{-1} \left[ 1 + \frac{1}{\epsilon E} \frac{\partial \epsilon}{\partial E} (\vec{E} \cdot \hat{k})^2 \right] \quad (27)$$

where  $\hat{k}$  is an unit vector in the  $\vec{K}$  direction:  $\hat{k} = \vec{K}/|\vec{K}|$ .

In the particular case of homogeneous and isotropic media, where  $\epsilon$  and  $\mu$  do not depend on the field strength, the effective geometry reduces to the Gordon [13] metric  $g^{\mu\nu} \equiv \eta^{\mu\nu} + (\epsilon\mu - 1)V^\mu V^\nu$ . The light paths will be determined by  $ds^2 = 0$ , where  $ds^2 = (1/\epsilon\mu)dt^2 - dx^2 - dy^2 - dz^2$ . It follows that the phase velocity associated to the electromagnetic wave is given, as expected, by  $v^2 = 1/\epsilon\mu$ , and the refraction index of the medium  $n_0 = \sqrt{\epsilon\mu}$ . The same result can be obtained directly from the general expression (27) by imposing  $\epsilon$  constant.

## V. THE EFFECTIVE GEOMETRY FOR ARTIFICIAL BIREFRINGENCE CAUSED BY AN EXTERNAL ELECTRIC FIELD

When an optically isotropic dielectric is submitted to an external electric field it is experimentally known that it may become optically anisotropic – the so called Kerr electro-optic effect. Under such situation the dielectric behaves optically like a uniaxial crystal with its optic axis parallel to the field direction [14]. Let us analyze this effect using the formalism developed before and derive the effective geometry associated to the wave propagation.

We set the following relation between the dielectric tensor  $\varepsilon_{\mu\nu}$  and the external field

$$\varepsilon^{\mu\beta} = \epsilon (\eta^{\mu\beta} - V^\mu V^\beta) - \alpha E^\mu E^\beta \quad (28)$$

with  $\epsilon$  and  $\alpha$  constants. For this case, taking the product of equation (18) with  $E_\mu$ , and by considering the relations (13) and (17), we obtain the light cone condition:

$$K^2 = [1 - \mu(\epsilon + 3\alpha E^2)] (KV)^2 + \frac{2\alpha}{(\epsilon + \alpha E^2)} E^\mu E^\nu K_\mu K_\nu. \quad (29)$$

Indeed, such condition can be expressed as  $g^{\mu\nu} K_\mu K_\nu = 0$ , where we have introduced the effective geometry

$$g^{\mu\nu} = \eta^{\mu\nu} + [\mu(\epsilon + 3\alpha E^2) - 1] V^\mu V^\nu - \frac{2\alpha}{(\epsilon + \alpha E^2)} E^\mu E^\nu. \quad (30)$$

The discontinuities of the electromagnetic field inside a dielectric that reacts to the action of an external field as stated by equation (28), propagates along null geodesics of this effective geometry.

The phase velocity yields

$$v^2 = \frac{1}{\mu(\epsilon + 3\alpha E^2)} \left[ 1 + \frac{2\alpha}{\epsilon + \alpha E^2} (\vec{E} \cdot \hat{k})^2 \right]. \quad (31)$$

As before,  $\hat{k}$  is an unit vector in the  $\vec{K}$  direction. We shall recognize two limiting cases from such expression. For  $\vec{E} \cdot \hat{k} = 0$ , which leads to  $v_\perp^2 = 1/\mu(\epsilon + 3\alpha E^2)$ ; and for  $\vec{E} \cdot \hat{k} = |\vec{E}|$ , which leads to  $v_\parallel^2 = 1/\mu(\epsilon + \alpha E^2)$ .

In order to identify the birefringence phenomenon we shall examine the eigen-vector problem stated by equation (18). Assuming the initial condition  $\vec{E} \cdot \vec{K} = 0$  we obtain

$$\{[\mu(KV)^2(\epsilon + \alpha E^2) + K^2 - (KV)^2] \eta^{\mu\nu} - 2\alpha\mu(KV)^2 E^\mu E^\nu\} e_\nu = 0. \quad (32)$$

The solutions for this eigen-vector problem can be achieved by expanding  $e_\mu$  as a linear combination of the linearly independent vectors  $E_\mu$ ,  $H_\mu$ ,  $K_\mu$  and  $V_\mu$ , resulting in the two light cone conditions on the wave propagation,

$$K^2 - (KV)^2 [1 - \mu(\epsilon + 3\alpha E^2)] = 0 \quad (33)$$

$$K^2 - (KV)^2 [1 - \mu(\epsilon + \alpha E^2)] = 0. \quad (34)$$

As we see, from equation (29), the condition (33) is already expected. Therefore, we also obtain the second solution (34) that corresponds to another mode of light propagation. Performing the computation of the phase velocity associated to this mode we obtain  $v^2 = 1/\mu(\epsilon + \alpha E^2)$ , which is the wave existing in the case where  $\vec{E} \cdot \hat{k} = |\vec{E}|$  from equation (29). Such mode survives for all directions of propagation, and there will be an additional mode depending on the angle from the wave propagation to the direction of the external electric field. This mode is governed by the effective geometry introduced by (30). Thus, birefringence occurs when an optically isotropic dielectric is submitted to an external electric field.

The effective geometries that govern the electromagnetic wave inside the medium will be given by (30), for the ray whose velocity depends on the direction of the external field, and by  $g^{\mu\nu} = \eta^{\mu\nu} + [\mu(\epsilon + \alpha E^2) - 1] V^\mu V^\nu$ , for the ray whose velocity does not depend on the direction of the external field.

For the limiting case, where  $\vec{E} \cdot \vec{K} = 0$ , the phase velocities are determined by  $v_\perp^2$  and  $v_\parallel^2$ , as calculated before and the index of refraction yields  $n_\perp^2 = \mu(\epsilon + 3\alpha E^2)$  and  $n_\parallel^2 = \mu(\epsilon + \alpha E^2)$ , respectively. In most cases the term in  $\alpha$  represents only a small correction of the values of  $\epsilon$ . Thus for electromagnetic waves propagating inside a material medium in a perpendicular direction to that defined by the external electric field, the difference between the index of refraction of the two existing rays yields

$$n_\perp - n_\parallel = n_0 \left[ \frac{\alpha E^2}{\epsilon} - \frac{\alpha^2 E^4}{\epsilon^2} + \mathcal{O}(\alpha^3) \right] \quad (35)$$

where  $n_0 = (\epsilon\mu)^{1/2}$  is the index of refraction of the isotropic case, as derived in the previous section. Finally when  $\alpha = 0$  we recover the isotropic case.

## VI. CONCLUSION

In this paper we have dealt with electromagnetic wave propagation inside a material medium. The general field equations were presented in terms of the field strengths  $(E^\mu, H^\mu)$  and the tensors  $\varepsilon_{\mu\nu}$  and  $\mu_{\mu\nu}$  that characterize the properties of each medium where the propagation occurs. By using the Hadamard method we derived the set of equations that govern the wave propagation. We examined some special cases where the light cone conditions can be obtained without solving the general Fresnel equation and for these cases we presented the associated effective geometry. As an application of the formalism, we analyzed the isotropic media and the artificial birefringence caused by an external electric field applied over a dielectric medium that reacts nonlinearly to the field excitation. For this case we obtained as a first order of approximation the known result that the difference between the index of refraction of the two limiting waves propagating in such dielectric is proportional to the square of the external field.

The method developed here can be applied to study the electromagnetic wave propagation inside material media. (We do not consider the frequency dependent material properties.) To do this we must know the expression for the tensors  $\varepsilon_{\mu\nu}$  and  $\mu_{\mu\nu}$  in terms of both the specific medium and the external field.

An effective geometry can be derived for each situation and can be used in the study of the properties of light propagation. With such geometrical description, we present tools for testing kinematic aspects of gravitation in laboratory, an issue very much addressed these days (see Refs. [15–18]). For instance, we could ask about the possibility of formation of structures like event horizons, bending of light and others, in laboratory.

A possible continuation of this work would be the comparison between the Lagrangian method resumed in the effective geometry stated by (1) and that obtained for each case that can be derived for non-linear media. Also deserves further investigation the general effective geometry associated with an arbitrary material medium in terms of the tensors  $\varepsilon_{\mu\nu}$  and  $\mu_{\mu\nu}$ .

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